MOMENTS OF INERTIA.

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It is the purpose of this paper to put on record formulæ for the Moments of Inertia of the plane areas, $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}=1$, and the solid bounded by the surface, $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}+\left(\frac{z}{c}\right)^{\frac{2}{2p+1}}=1$.

Let μ be the mass of a unit, (a) area, (b) volume.

(a) Areas, when n and m are positive integers.

For the x-axis,

$$I=4\mu\int\int y^2\,dxdy=\frac{4ab^3\mu}{\frac{4}{(2m+1)(2n+1)}}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{3}{2})}{\Gamma(m+3n+3)}$$

$$=\frac{\mu ab^3(2m+1)(2n+1)(6n+1)\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{1}{2})}{2(m+3n+2)(m+3n)(m+3n+1)\Gamma(m+3n)}\text{...... ......}(1)$$

$$=\frac{1.3.5\ldots\ldots(2m+1)\times1.3.5\ldots\ldots(6n+1)}{2.4.6\ldots\ldots2(m+3n+2)}\cdot 2\pi\mu ab^3(2n+1)\ \ldots\ldots\ (2).$$

For the y-axis,

$$I_1=4\mu\int\int x^2\,dxdy=\frac{\mu a^3b(2m+1)(2n+1)(6m+1)\Gamma(3m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{2(3m+n+2)(3m+n+1)(3m+n)\Gamma(3m+n)}\ \ldots\ldots..(3)$$

$$=\frac{1.3.5\ldots\ldots(6m+1)\times1.3.5\ldots\ldots(2n+1)}{2.4.6\ldots\ldots2(3m+n+2)}\cdot 2\pi\mu a^3b(2m+1)\ldots.\ \ (4).$$

For an axis through its center perpendicular to its plane,

$$I_2=I+I_1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The product of inertia of a quadrant about its axes is,

$$p=\mu\int\int xy\,dxdy=\frac{\mu a^2b^2}{\frac{4}{(2m+1)(2n+1)}}\cdot\frac{\Gamma(2m+2)\Gamma(2n+1)}{\Gamma(2m+2n+3)}$$

$$=\frac{\mu a^2b^2mn(2m+1)(2n+1)\Gamma(2m)\Gamma(2n)}{4(m+n+1)(m+n)(2m+2n+1)\Gamma(2m+2n)}\ \ldots\ \ldots\ldots\ldots(6)$$

$$=\frac{1.2.3.4\ldots\ldots(2m+1)\times 1.2.3.4\ldots\ldots(2n+1)}{1.2.3.4\ldots\ldots(2m+2n+2)}\qquad\frac{\mu a^2b^2}{4}\ \ldots\ldots\ldots(7).$$

Let $m=n=0$. Then for the ellipse, $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1.$

$$I=\tfrac{1}{4}\pi\mu ab^3,\quad I_1=\tfrac{1}{4}\pi\mu a^3b,\quad I_2=\tfrac{1}{4}\pi\mu ab(a^2+b^2),\quad p=\tfrac{1}{8}\mu a^2b^2.$$

Let $m=n=1$. Then for the hypocycloid, $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$I=\tfrac{21}{512}\pi\mu ab^3,\quad I_1=\tfrac{21}{512}\pi\mu a^3b,\quad I_2=\tfrac{21}{512}\pi\mu ab(a^2+b^2),\quad p=\tfrac{1}{80}\mu a^2b^2.$$

(b) Solids, when m, n, and p are positive integers.

With regard to the plane (yz),

$$\mathbf{I}=8\mu\int\!\!\int\!\!\int x^2\,dxdydz$$

$$=\frac{8\mu a^3bc}{\dfrac{8}{(2m+1)(2n+1)(2p+1)}}\cdot\frac{\Gamma(3m+\frac{3}{2})\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{\Gamma(3m+n+p+\frac{5}{2})}$$

$$=\frac{4\mu a^3bc(2m+1)(2n+1)(2p+1)(6m+1)\Gamma(3m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{(6m+2n+2p+5)(6m+2n+2p+3)(6m+2n+2p+1)\Gamma(3m+n+p+\frac{1}{2})}\ \ldots(8)$$

$$=\frac{1.3.5\ldots\ldots(6m+1)\times 1.3.5\ldots\ldots(2n+1)\times 1.3.5\ldots\ldots(2p+1)}{1.3.5\ldots\ldots(6m+2n+2p+5)}$$

$$\times 4\mu\pi a^3bc(2m+1)\ldots\ldots(9).$$

With regard to the plane (xz),

$$\mathbf{I}_1=8\mu\int\!\!\int\!\!\int y^2\,dxdydz$$

$$=\frac{4\mu ab^3c(2m+1)(2n+1)(2p+1)(6n+1)\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{(2m+6n+2p+5)(2m+6n+2p+3)(2m+6n+2p+1)\Gamma(m+3n+p+\frac{1}{2})}\ \ldots(10)$$

$$=\frac{1.3.5\ldots\ldots(2m+1)\times 1.3.5\ldots\ldots(6n+1)\times 1.3.5\ldots\ldots(2p+1)}{1.3.5\ldots\ldots(2m+6n+2p+5)}$$

$$\times 4\mu\pi ab^3c(2n+1)\ldots\ldots(11).$$

With regard to the plane (xy),

$$\mathbf{I}_2=8\mu\int\!\!\int\!\!\int z^2\,dxdydz$$

$$=\frac{4\mu abc^3(2m+1)(2n+1)(2p+1)(6p+1)\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(3p+\frac{1}{2})}{(2m+2n+6p+5)(2m+2n+6p+3)(2m+2n+6p+1)\Gamma(m+n+3p+\frac{1}{2})}\ \ldots(12)$$

$$=\frac{1.3.5\ldots\ldots(2m+1)\times1.3.5\ldots\ldots(2n+1)\times1.3.5\ldots\ldots(6p+1)}{1.3.5\ldots\ldots(2m+2n+6p+5)}$$

$$\times 4\mu\pi abc^3(2p+1)\ldots\ldots(13).$$

$\mathbf{I}_3=\mathbf{I}+\mathbf{I}_1$, for z-axis, $\mathbf{I}_4=\mathbf{I}+\mathbf{I}_2$, for y-axis,

$\mathbf{I}_5=\mathbf{I}_1+\mathbf{I}_2$, for x-axis, $\mathbf{I}_6=\mathbf{I}+\mathbf{I}_1+\mathbf{I}_2$, for center.

Product of inertia of an octant of the solid with regard to the (y, z) axes,

$$P=\mu\int\int\int yzdxdydz=\frac{\frac{\mu ab^2c^2}{8}}{(2m+1)(2n+1)(2p+1)}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(2n+1)\Gamma(2p+1)}{\Gamma(m+2n+2p+\frac{5}{2})}.$$

$$=\frac{4\mu ab^2c^2np(2m+1)(2n+1)(2p+1)\Gamma(m+\frac{1}{2})\Gamma(2n)\Gamma(2p)}{(2m+4n+4p+5)(2m+4n+4p+3)(2m+4n+4p+1)\Gamma(m+2n+2p+\frac{1}{2})}\ldots(14)$$

$$=\frac{1.2.3\ldots\ldots(2n+1)\times1.2.3\ldots\ldots(2p+1)\times\frac{1}{2}.\frac{3}{2}.\frac{5}{2}\ldots\ldots\left(\frac{2m+1}{2}\right)}{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}\ldots\ldots\left(\frac{2m+4n+4p+5}{2}\right)}\cdot\frac{\mu ab^2c^2}{4}$$

$$=\frac{1.2.3.4\ldots\ldots(2n+1)\times1.2.3.4\ldots\ldots(2p+1)}{(2m+3)(2m+5)\ldots\ldots(2m+4n+4p+5)}\cdot\mu ab^2c^2.2^{2(n+p)}\ldots(15).$$

With regard to the axes (x, z),

$$P_1=\mu\int\int\int xzdxdydz$$

$$=\frac{4\mu a^2bc^2mp(2m+1)(2n+1)(2p+1)\Gamma(2m)\Gamma(n+\frac{1}{2})\Gamma(2p)}{(4m+2n+4p+5)(4m+2n+4p+3)(4m+2n+4p+1)\Gamma(2m+n+2p+\frac{1}{2})}\ldots(16)$$

$$=\frac{2^{2(m+p)}1.2.3.4\ldots\ldots(2m+1)\times1.2.3.4\ldots\ldots(2p+1)}{(2n+3)(2n+5)\ldots\ldots(4m+2n+4p+5)}\mu a^2bc^2\ldots(17).$$

With regard to the axes (x, y),

$$P_2=\mu\int\int\int xydxdydz$$

$$=\frac{4\mu a^2b^2cmn(2m+1)(2n+1)(2p+1)\Gamma(2m)\Gamma(2n)\Gamma(p+\frac{1}{2})}{(4m+4n+2p+5)(4m+4n+2p+3)(4m+4n+2p+1)\Gamma(2m+2n+p+\frac{1}{2})}\ldots(18)$$

$$=\frac{2^{2(m+n)}1.2.3.4\ldots\ldots(2m+1)\times1.2.3.4\ldots\ldots(2n+1)}{(2p+3)(2p+5)\ldots\ldots(4m+4n+2p+5)}\cdot\mu a^2b^2c\ldots(19).$$

Let $m=n=p=0$. Then for $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$,

$\mathbf{I}=\frac{4}{15}\mu\pi a^3bc$, $\mathbf{I}_1=\frac{4}{15}\mu\pi ab^3c$, $\mathbf{I}_2=\frac{4}{15}\mu\pi abc^3$,

$\mathbf{I}_3=\frac{4}{15}\mu\pi abc(a^2+b^2)$, $\mathbf{I}_4=\frac{4}{15}\mu\pi abc(a^2+c^2)$,

$\mathbf{I}_5=\frac{4}{15}\mu\pi abc(b^2+c^2)$, $\mathbf{I}_6=\frac{4}{15}\mu\pi abc(a^2+b^2+c^2)$,

$P=\frac{1}{15}\mu ab^2c^2$, $P_1=\frac{1}{15}\mu a^2bc^2$, $P_2=\frac{1}{15}\mu a^2b^2c$.

Let $m=n=p=1$. Then for $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$,

$\mathbf{I}=\frac{4}{715}\mu\pi a^3bc$, $\mathbf{I}_1=\frac{4}{715}\mu\pi ab^3c$, $\mathbf{I}_2=\frac{4}{715}\mu\pi abc^3$,

$\mathbf{I}_3=\frac{4}{715}\mu\pi abc(a^2+b^2)$, $\mathbf{I}_4=\frac{4}{715}\mu\pi abc(a^2+c^2)$,

$\mathbf{I}_5=\frac{4}{715}\mu\pi abc(b^2+c^2)$, $\mathbf{I}_6=\frac{4}{715}\mu\pi abc(a^2+b^2+c^2)$,

$P=\dfrac{64\mu ab^2c^2}{15.13.11.7.5}$, $P_1=\dfrac{64\mu a^2bc^2}{15.13.11.7.5}$, $P_2=\dfrac{64\mu a^2b^2c}{15.13.11.7.5}$.

Thus we could multiply examples without number.

Formulæ (1), (3), (6), (8), (10), (12), (14), (16), (18), will hold for m, n, p fractional as well as integral.

For the radius of gyration we have

$$k_{\prime\prime}^2=\frac{I_{\prime\prime}}{\mu A}, \quad K_{\prime\prime}^2=\frac{\mathbf{I}_{\prime\prime}}{\mu V},$$

where A and V are known, (see American Mathematical Monthly, page 380, Vol. I., No. 11.)

A SIMPLE DEDUCTION OF THE DIFFERENTIAL OF LOGx.

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Let $f(x)=\log x$ ....(1), then $f(xy)=f(x)+f(y)$ ....(2).

Differentiate, $f'(xy)(ydx+xdy)=f'(x)dx+f'(y)dy$ ....(3).

Since (3) is true when x and y are independent,

$f'(xy)ydx=f'(x)dx$ ....(4), and $f'(xy)xdy=f'(y)dy$. ....(5).

(4)$\div$(5), $\dfrac{f'(x)}{f'(y)}=\dfrac{y}{x}=\dfrac{1/x}{1/y}$ ....(6).

$\therefore f'(x)=\dfrac{m}{x}$, $f'(y)=\dfrac{m}{y}$ ....(7). $\therefore d\log x=\dfrac{m}{x}dx$ ....(8).